Constructing and (some) classification of integer matrices with integer eigenvalues

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Solve a linear system:

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$$\mathbf{x} = C_1\mathbf{v}_1 e^{\lambda_1 t} + C_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + C_n\mathbf{v}_n e^{\lambda_n t}$$

... or something similar.

An example:

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

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Reasonable?

GOAL: Construct a matrix A with relatively small *integer* entries and with relatively small *integer* eigenvalues. (IMIE)

Such an A will be a *good example*.

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Note: Martin and Wong, "Almost all integer matrices have no integer eigenvalues"

Recall/notation:

$$A = PDP^{-1}$$

where

$$P = \begin{pmatrix} | & | & \cdots & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & \cdots & | \end{pmatrix}$$

has columns which form a basis of eigenvectors for A and $D = \langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$ is the diagonal matrix with corresponding eigenvalues on the diagonal.

Our example:

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$D = \langle 1, 1, 5 \rangle$$

And in this case,

$$P = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A = PDP^{-1}.$$

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Then

$$P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} P^{-1} = \begin{pmatrix} 5/2 & 3 & -1/2 \\ 1 & 3 & 1 \\ 1/2 & 1 & 7/2 \end{pmatrix}$$

The problem is that

$$P^{-1} = (1/4)\begin{pmatrix} -1 & 2 & -1 \\ -1 & -2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

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$$P^{-1} = (1/4)\begin{pmatrix} -1 & 2 & -1 \\ -1 & -2 & 3 \\ 1 & 2 & 1 \end{pmatrix} = \frac{1}{\det P} P^{\text{adj}}$$

Or, really, that $\det P = 4$.

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But actually ...

Proposition (Eigenvalues congruent to b modulo k)

Let P be an $n \times n$ invertible matrix with $k = \det P \neq 0$. Suppose every $\lambda_i \equiv b \mod k$. (So all the λ_i's are congruent to each other.) Then $A = P\langle\lambda_1, \ldots, \lambda_n\rangle P^{-1}$ is integral.

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Good?

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Input: eigenvectors (really P)
Select good eigenvalues
Output: IMIE and its characteristic polynomial

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Idea [Renaud, 1983]: Take X, Y integer matrices. If YX is upper (or lower) triangular then the eigenvalues are the diagonal elements, hence integers. So XY will be a good example.

If X and Y are $n \times n$, we are imposing $n(n-1)/2$ othogonality conditions on the rows of Y and columns of X.

Let $X = \begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$.

Then $XY = \begin{pmatrix} 7 & 4 \\ 3 & -4 \end{pmatrix}$ has the same eigenvalues as

$YX = \begin{pmatrix} -5 & * \\ 0 & 8 \end{pmatrix}$, namely $\lambda = -5, 8$.

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$YX = \begin{pmatrix} -5 & * \\ 0 & 8 \end{pmatrix}$, namely $\lambda = -5, 8$.

Starting with X, the only condition on Y is that the second row of Y must be orthogonal to the first row of X.

Better fact:

Theorem (Folk Theorem)

Let U be an $r \times s$ matrix and V be an $s \times r$ matrix, where $r \leq s$. Then $\chi_{UV}(x) = x^{s-r}\chi_{VU}(x)$.

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Proof idea (due to Horn and Johnson):

$$\begin{pmatrix} UV & 0 \\ V & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ V & VU \end{pmatrix}$$

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via $\begin{pmatrix} I_r & U \\ 0 & I_s \end{pmatrix}$. □

In other words, UV and VU have nearly the same characteristic polynomials and nearly the same eigenvalues.

The only difference: 0 is a eigenvalue of the larger matrix, repeated as necessary.

For now, let us take U to be $n \times (n-1)$ and V to be $(n-1) \times n$.

An example. Begin with

$$U = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & -1 & 1 \end{pmatrix}$$

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$$VU = \begin{pmatrix} 2 & 0 & -4 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

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and

$$UV = \begin{pmatrix} -4 & -10 & 6 & 0 \\ 1 & 3 & -1 & 3 \\ 0 & -2 & 2 & 4 \\ 4 & 6 & -4 & 0 \end{pmatrix}.$$

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So UV is an integer matrix with eigenvalues $\lambda = 2, -3, 2$ and $\lambda = 0$.

Writing

$$U = \begin{pmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{pmatrix}$$

$$V = \begin{pmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_2 & - \\ - & \vec{v}_3 & - \end{pmatrix}$$

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We needed $\vec{v}_2 \cdot \vec{u}_1 = \vec{v}_3 \cdot \vec{u}_1 = \vec{v}_3 \cdot \vec{u}_2 = 0$.

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Then the eigenvalues of UV are the three dot products $\vec{v}_i \cdot \vec{u}_i$ and 0.

Note: we now only need an $(n-1) \times (n-1)$ matrix to be triangular. There are now $(n-1)(n-2)/2$ orthogonality conditions.

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This means there are *no* conditions to be checked in order to construct a 2×2 example.

Any

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}$$

is an integer matrix with eigenvalues 0 and $u_1 v_1 + u_2 v_2$.

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is an integer matrix with eigenvalues 0 and $u_1 v_1 + u_2 v_2$.

E.g.

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix}$$

$\lambda = -2, 0$

Disadvantages:

- We always get 0 as an eigenvalue.
- What are the eigenvectors?

Advantage:

- This is (nearly) a complete characterization for these matrices.

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Why does this work? Note that A and $A + bI$ have the same eigenvectors.

Example:

$$VU = \begin{pmatrix} 2 & 0 & -4 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{so} \quad UV = \begin{pmatrix} -4 & -10 & 6 & 0 \\ 1 & 3 & -1 & 3 \\ 0 & -2 & 2 & 4 \\ 4 & 6 & -4 & 0 \end{pmatrix}.$$

has eigenvalues $2, -3, 2, 0$.

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So

$$A = UV - I = \begin{pmatrix} -5 & -10 & 6 & 0 \\ 1 & 2 & -1 & 3 \\ 0 & -2 & 1 & 4 \\ 4 & 6 & -4 & -1 \end{pmatrix}$$

has eigenvalues $1, -4, 1, -1$.

Earlier we had

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with $\lambda = -2, 0$
So

$$UV + I = \begin{pmatrix} 3 & -8 \\ 1 & -3 \end{pmatrix}$$

has eigenvalues $\lambda = -1, 1$.

Note: This gives a quick proof to our Eigenvalues Congruent to b modulo k Proposition.

This is actually a classification of all IMIEs.

Theorem (Renaud)

Every $n \times n$ integer matrix with integer eigenvalues can be written as a $UV + bI_n$ where VU is a triangular $(n-1) \times (n-1)$ matrix.

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Every $n \times n$ integer matrix with integer eigenvalues can be written as a $UV + bI_n$ where VU is a triangular $(n-1) \times (n-1)$ matrix.

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What if U is $n \times r$ and V is $r \times n$?

In particular, if U is $n \times 1$ and V is $1 \times n$ then VU is trivially triangular.

Take $U = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$.

Then $VU = (4)$ and UV must be a 3×3 integer matrix with eigenvalues $4, 0, 0$.

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In fact $UV + I_3 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} = A$, our original example.

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Suppose VU is not just triangular, but *diagonal*.

Then the $(n-1)$ columns of U are eigenvectors corresponding to the $\lambda_1, \ldots, \lambda_{n-1}$.
Any vector, $\vec{w}$ in the null space of V is an eigenvector corresponding to b.

Example. $U = \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

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So the columns of U are eigenvectors (corresponding to $\lambda = -4, -2$, resp.) of

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$\vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has $V\vec{w} = \vec{0}$. So it is an eigenvector for UV corresponding to $\lambda = 0$.

Recall: $A = PDP^{-1}$ approach.

Idea 1. Make sure P has determinant ± 1.

How to construct such P?

One technique [Ortega, 1984] uses:
Take $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then $P = I_n + \vec{u} \otimes \vec{v}$ has $\det P = 1 + \vec{u} \cdot \vec{v}$.

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This is just a special case of what we have been doing with the Folk Theorem and the Shift Propostion.

Example. Try $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

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Then

$$\vec{u} \otimes \vec{v} = \begin{pmatrix} -2 & -1 & 1 \\ -4 & -2 & 2 \\ -8 & -4 & 4 \end{pmatrix}$$

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Get

$$P = I + \vec{u} \otimes \vec{v} = \begin{pmatrix} -1 & -1 & 1 \\ -4 & -1 & 2 \\ -8 & -4 & 5 \end{pmatrix}.$$

and

$$P^{-1} = I - \vec{u} \otimes \vec{v} = \begin{pmatrix} 3 & 1 & -1 \\ 4 & 3 & -2 \\ 8 & 4 & -3 \end{pmatrix}.$$

So Ortega uses this to say that any $A = PDP^{-1}$ will be an IMIE.

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But in fact, P itself is an IMIE. It has the single eigenvalue 1.

P is clearly non-diagonalizable.

$$\begin{pmatrix} -1 & -1 & 1 \\ -4 & -1 & 2 \\ -8 & -4 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Ortega's full result is that if $\vec{u} \cdot \vec{v} = \beta$ then $P = I_n + \vec{u} \otimes \vec{v}$ has $\det P = 1 + \beta$ and (if $\beta \neq -1$) then $P^{-1} = I_n - \frac{1}{1+\beta} \vec{u} \otimes \vec{v}$.

The case $\beta = -2$ is interesting.

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with $\lambda = -2, 0$

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with $\lambda = -2, 0$
So

$$Q = UV + I = \begin{pmatrix} 3 & -8 \\ 1 & -3 \end{pmatrix}$$

has eigenvalues $\lambda = -1, 1$.
In fact, this Q is coming from Ortega's construction. So $Q^{-1} = Q$.

- Christopher Towse and Eric Campbell. *Constructing integer matrices with integer eigenvalues. The Math. Scientist* **41**(2): 45–52, 2016.
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