Finite Temporal Logic

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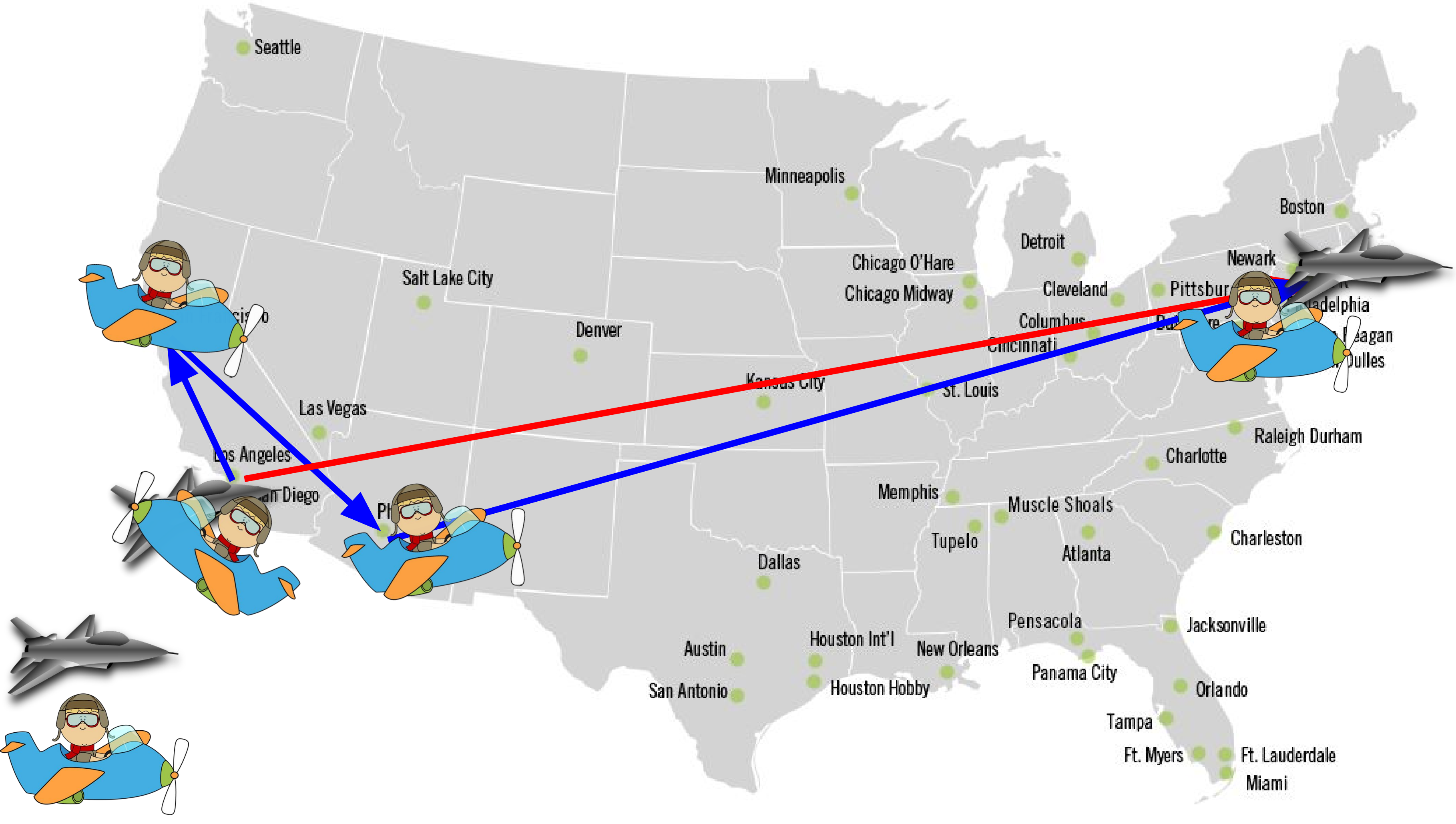
Advised by Michael Greenberg



http://www.williamsonchamber.com/wp-content/uploads/air-travel-map2.jpg

Questions:

- Do Speedy & Sleepy get to JFK?
- Do Speedy & Sleepy stop flying?
- Speedy has a warrant in Texas. Does Speedy Land in Texas?

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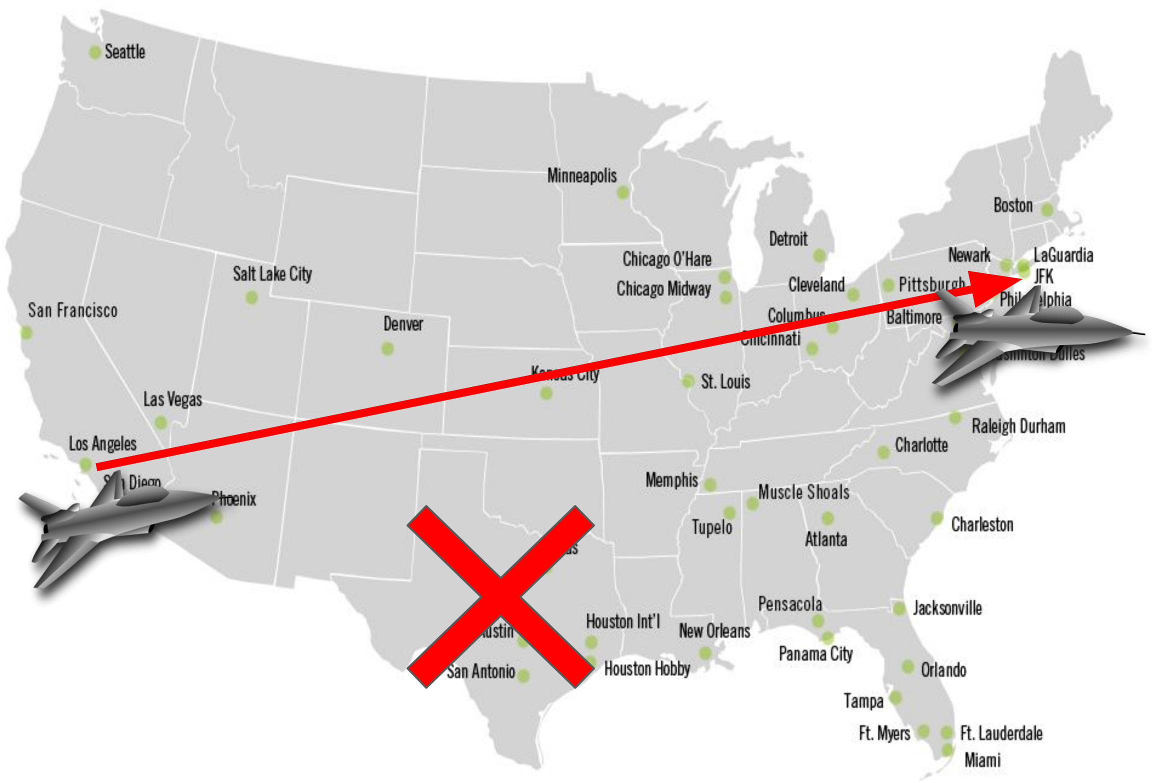

"True" Expressions

LAX

next(JFK)

ever(end), next(end)

always(¬ TX)

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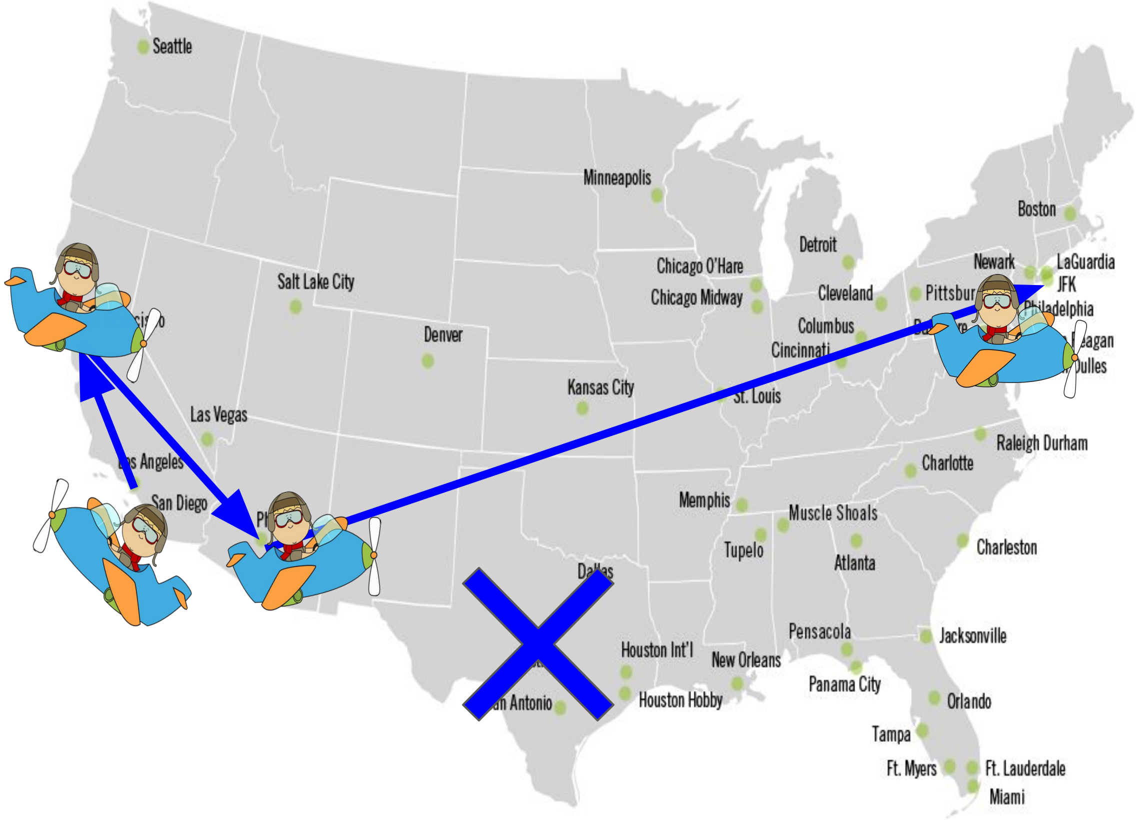

"True" Expressions

LAX

next(SFO)

next(next(PHX))

next(next(next(JFK)))

next(next(next(end)))

always(¬ TX)

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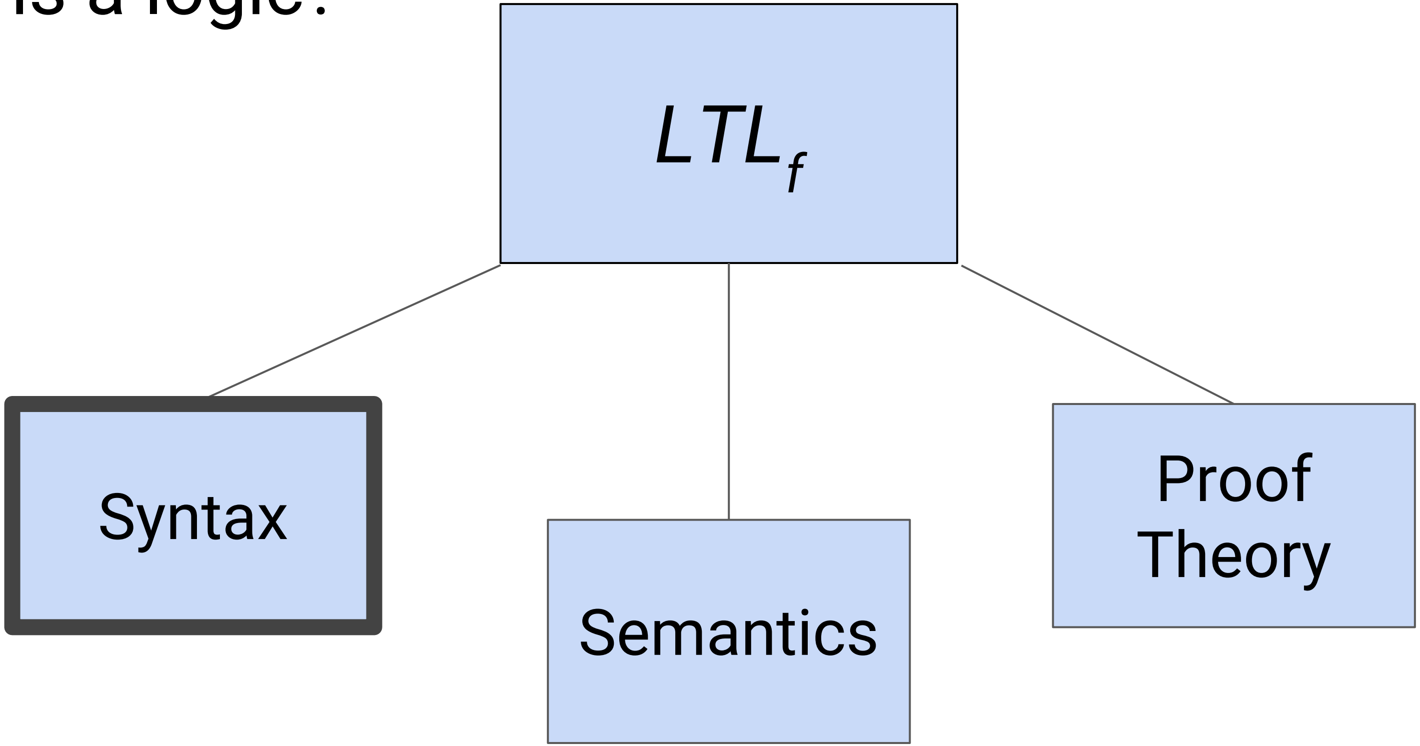
Linear Temporal Logic over finite traces (LTL_f)

What is a logic?



Syntax for LTL_f

Define the set of formulae, $LTL_f(\mathbf{V})$, to be

a,b ::= v | False | True | end
| ¬ a | next a | always a | ever a
| $a \rightarrow b$ | $a \vee b$ | $a \wedge b$

Or equivalently,

$LTL_f(\mathbf{V}) = \langle\ \mathbf{V} \cup$ {True, False, end},
{ ¬ , next, always, ever},
$\{\rightarrow, \vee, \wedge\}\ \rangle$

Example Formulae

V = {LAX, SFO, PHX, JFK,
CA, TX}

LAX, SFO, LAX → CA

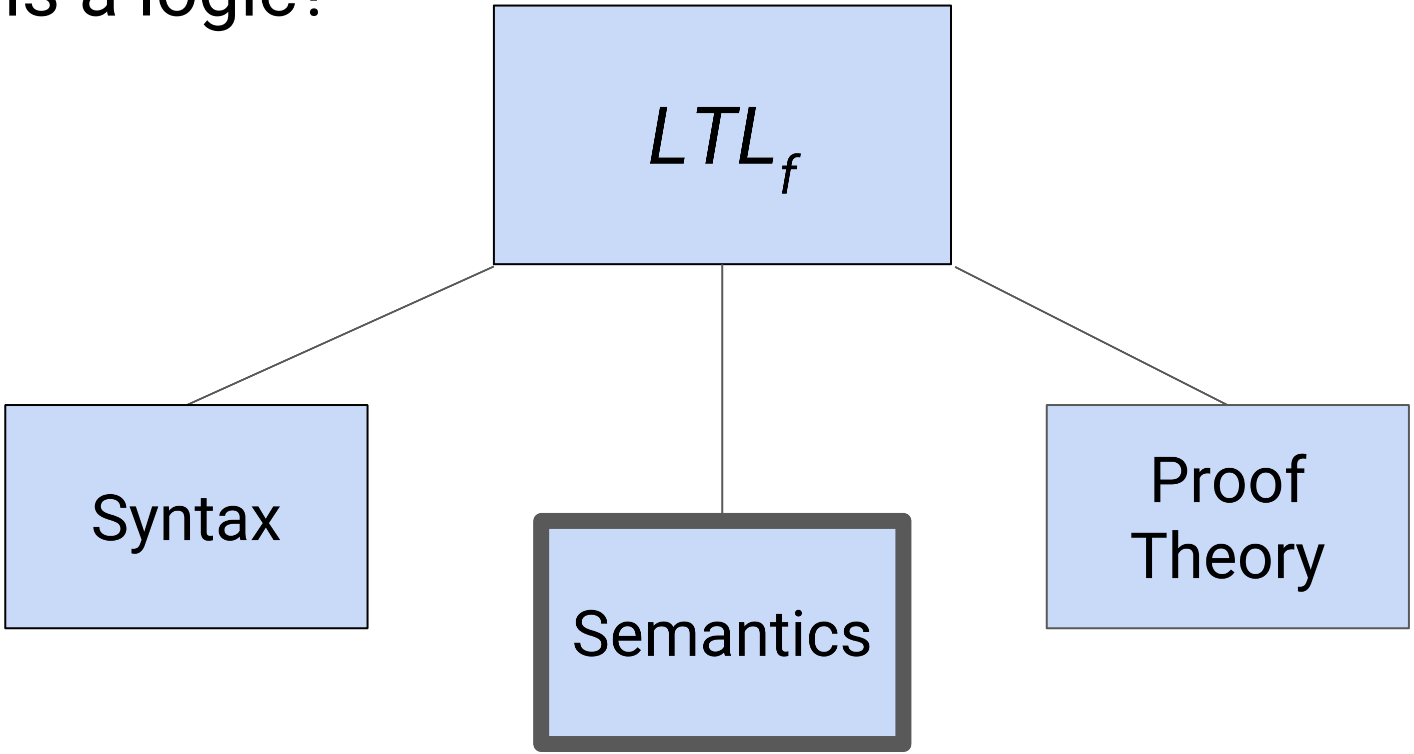
next(PHX)

ever(end)

ever(JFK ∧ end) → ¬ always(¬ JFK)

LAX → ever(JFK)

What is a logic?

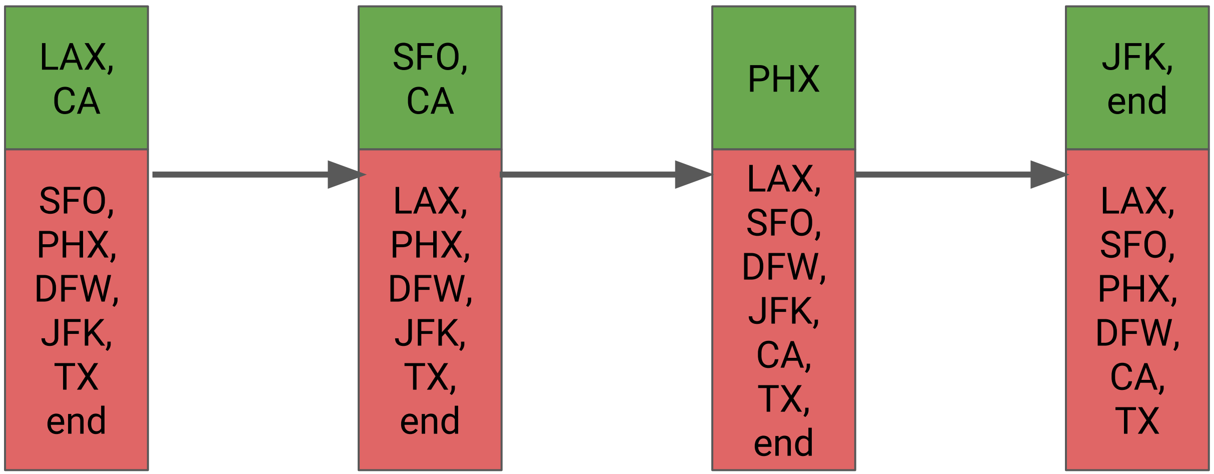


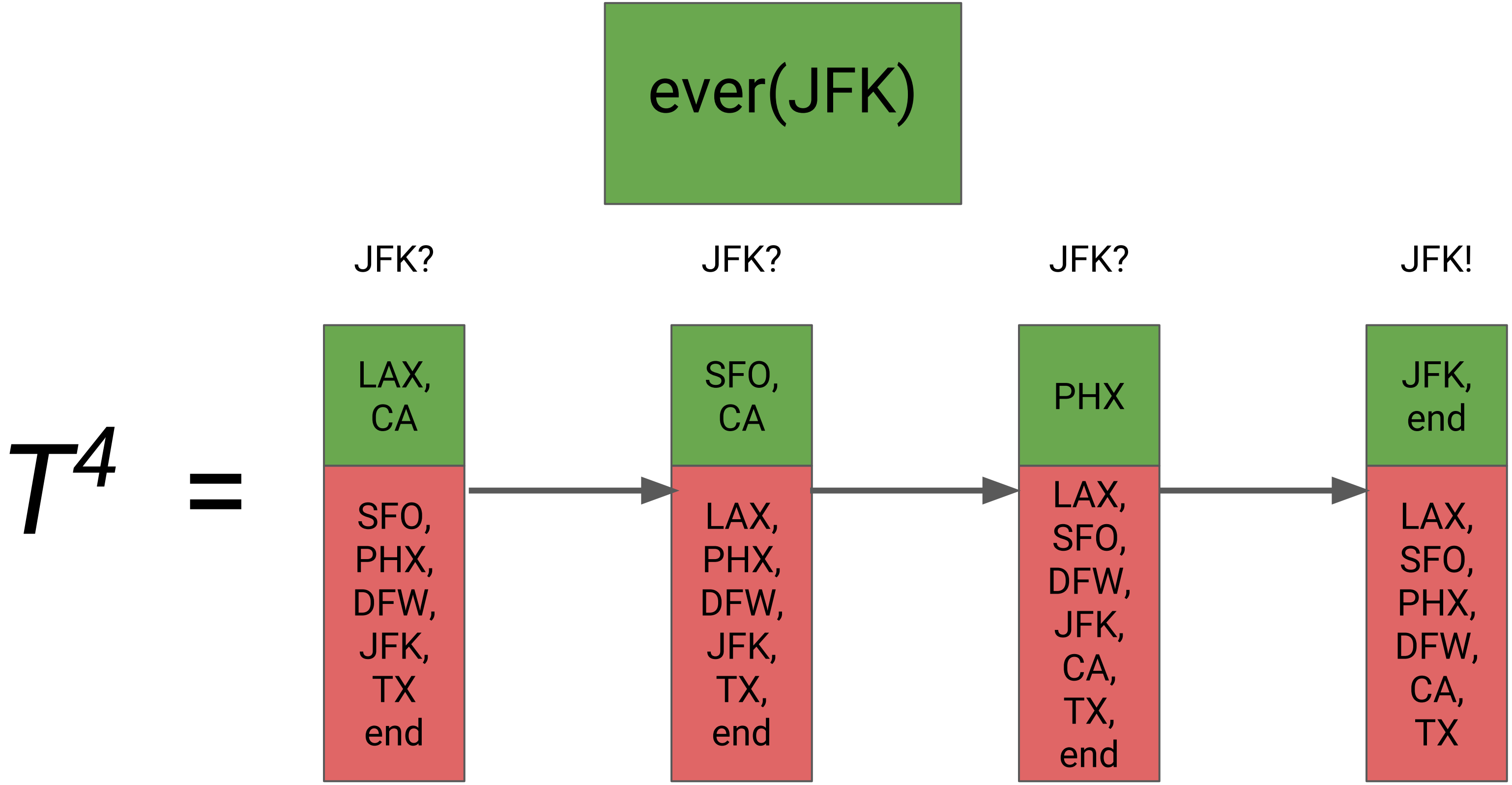
Semantics: A Ground Truth

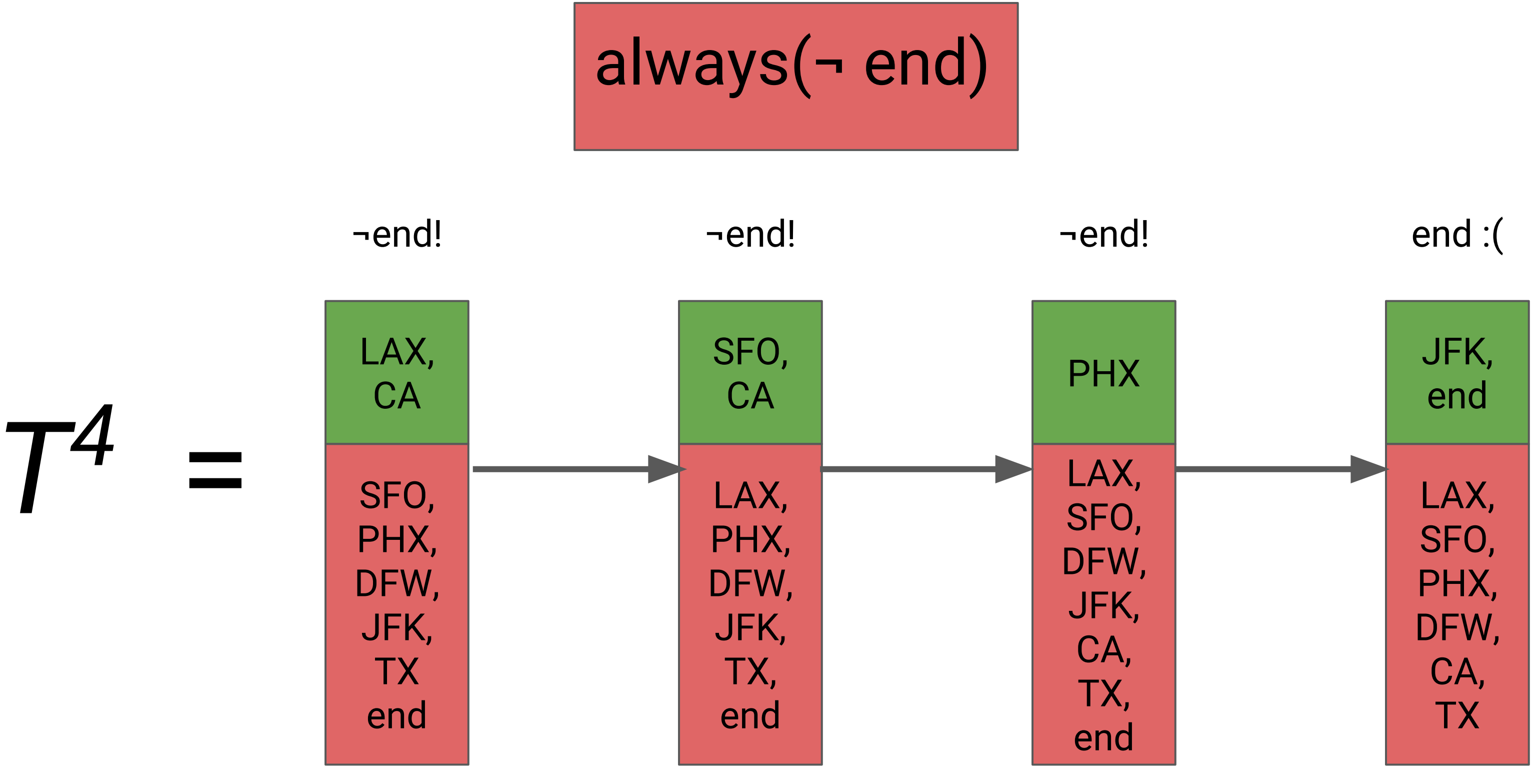
Definition: Timeline

A *timeline* $T^n = (f_1, f_2, \ldots, f_n)$ is an *n*-tuple of functions $f_i : \mathbf{V} \rightarrow \{$ T , F $\}$

$$T^4 =$$




ever(JFK)
JFK?
JFK?
JFK?
JFK!
T^4 =
LAX, CA
SFO, PHX, DFW, JFK, TX end
SFO, CA
LAX, PHX, DFW, JFK, TX, end
PHX
LAX, SFO, DFW, JFK, CA, TX, end
JFK, end
LAX, SFO, PHX, DFW, CA, TX



always(¬ end)
T^4 =
¬end!
¬end!
¬end!
end :(
LAX, CA
SFO, PHX, DFW, JFK, TX end
SFO, CA
LAX, PHX, DFW, JFK, TX, end
PHX
LAX, SFO, DFW, JFK, CA, TX, end
JFK, end
LAX, SFO, PHX, DFW, CA, TX

Semantics: Validity (⊨)

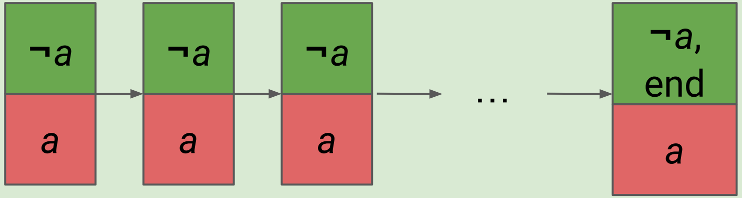
Definition: Validity

A formula *a* is *valid* if for every timeline T^n

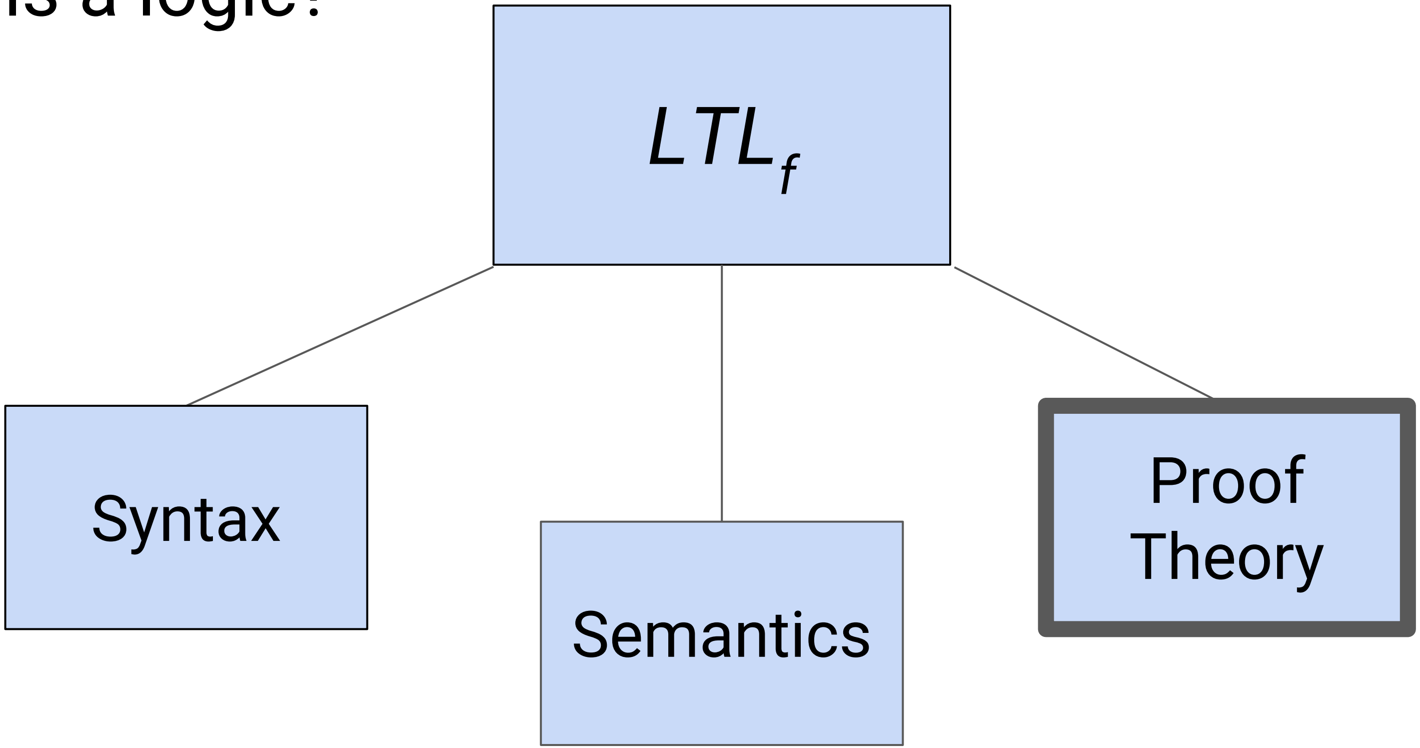
$T^n_i(a) = \text{True} \qquad \forall\ i = 1, \ldots, n,$

Write ⊨ a , if a is valid.

Example: ⊨ always(¬ *a*) → ¬ ever(*a*).



What is a logic?



Towards a Proof Theory

- Proofs for specific timelines are done by induction
- Syntactic proofs for timelines in general -- without Timelines??

We want to be able to prove *valid formulae*.

Tauts in Classical Logic

$\vdash$ next $(a \rightarrow b)$ $\equiv$ next$(a) \rightarrow$ next(b)

$\vdash$ end $\rightarrow \neg$ next(a)

$\vdash$ ever(end)

$\vdash$ ever a $\equiv$ $a \vee$ next(ever(a))

If $\vdash a$, then $\vdash$ wk_next(a)

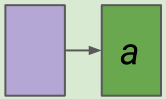
If $\vdash a \rightarrow b$

and $\vdash a \rightarrow$ wk_next(a),

then $\vdash a \rightarrow$ always(b)

No Reference to Timelines!!

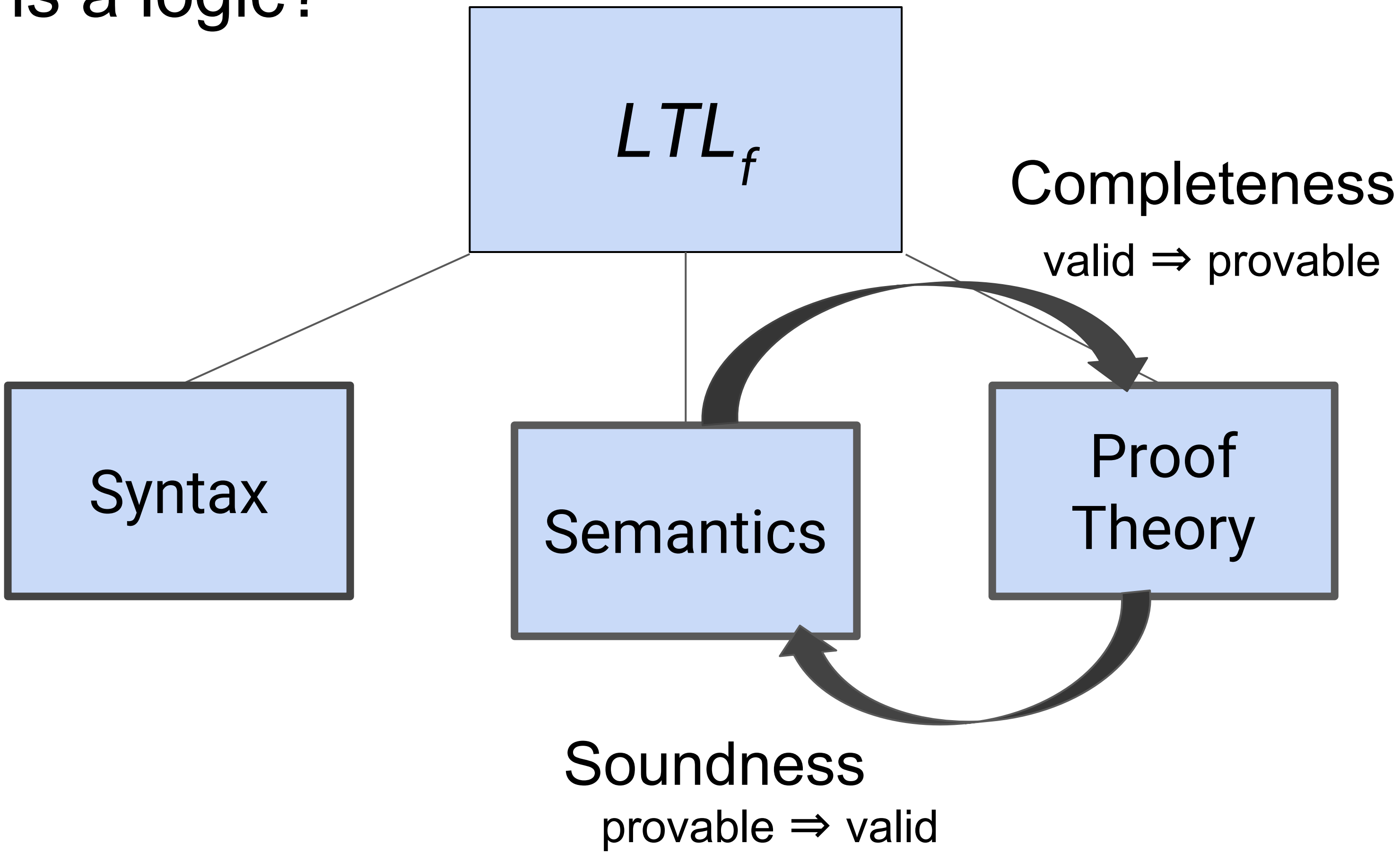
Example Proof ($\vdash$ next(a) $\rightarrow \neg$ end)



Proof.

1.	next(a)	Assume that next(a) holds
2.	end	To show ¬ end, assume end for contradiction
3.	¬ next(a)	Apply $\vdash$ end $\rightarrow \neg$ next(a) to end giving ¬ next(a)
4.	contradiction	¬ next(a) and next(a) are a contradiction
5.	¬ end	end was contradictory, so ¬ end ▲

What is a logic?



Soundness & Completeness

Theorem: Soundness

If $\vdash a$ then $\vDash a$.

If we can prove *a*, then *a* is valid.

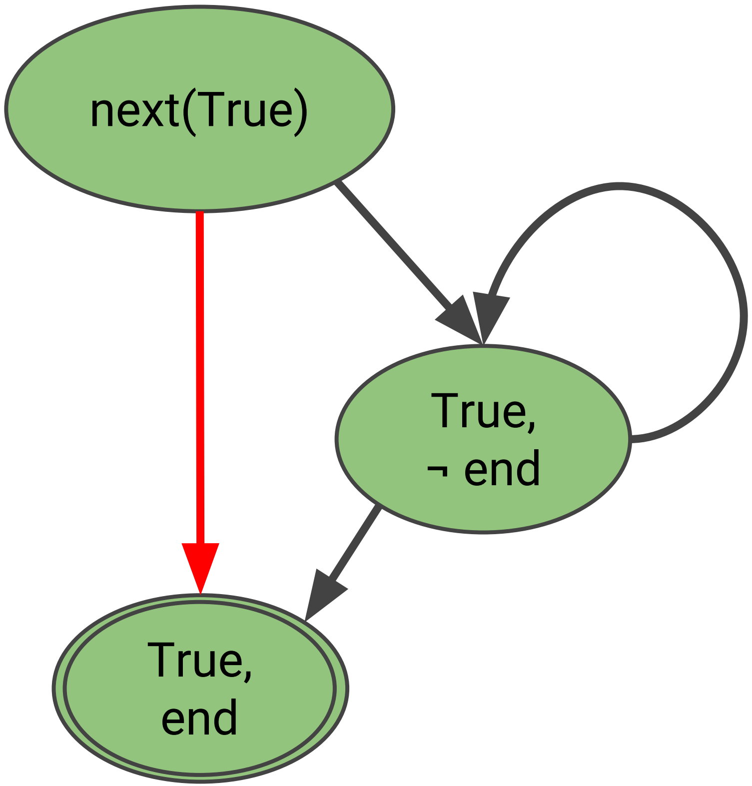
Proof. Simple. show all rules are valid. □

Theorem: Completeness

If $\vDash a$ then $\vdash a$.

If *a* is valid, we can prove *a*.

Proof. Hard! Construct a graph modelling to all possible timelines. □



One other property: Decidability

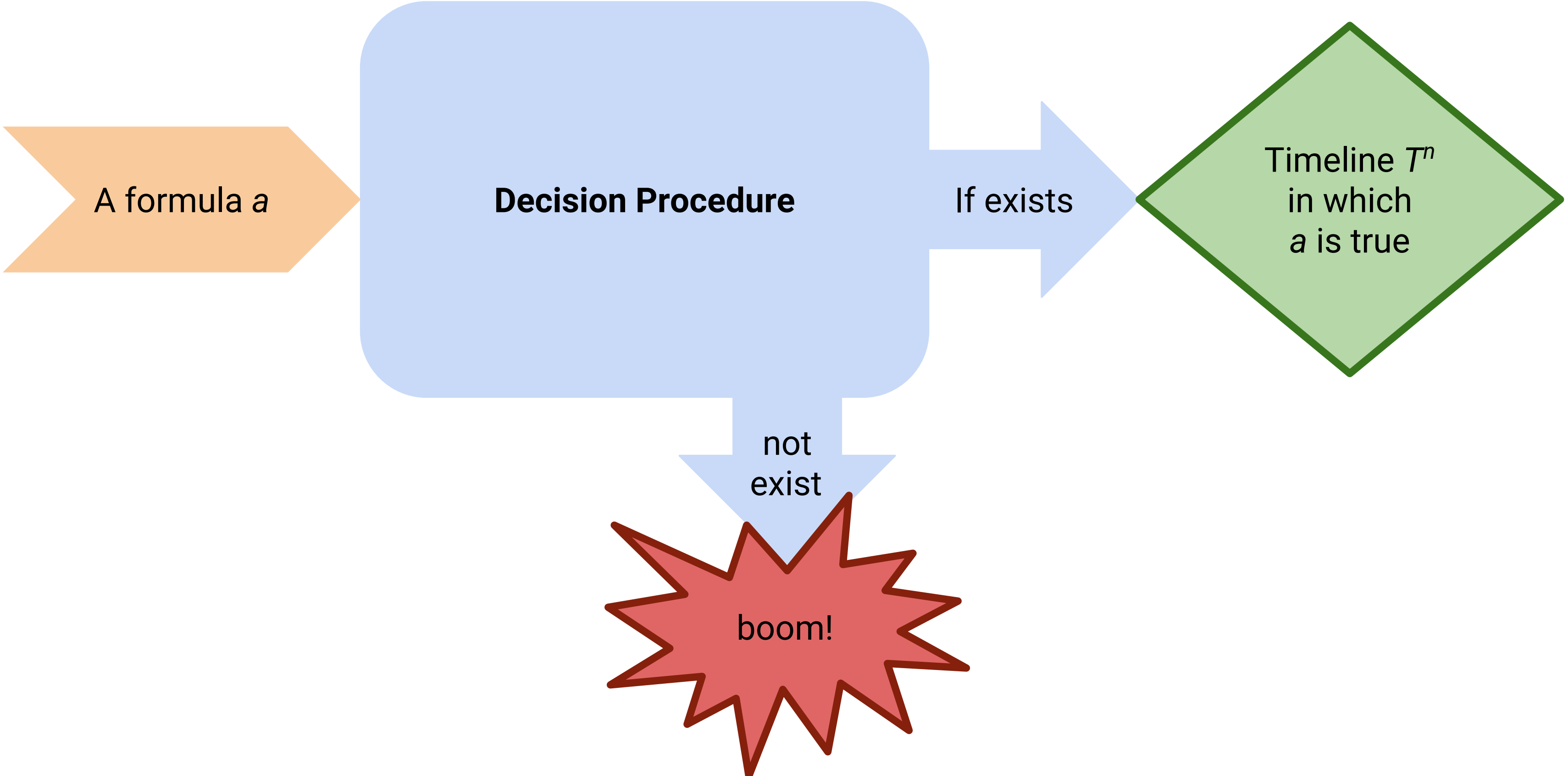
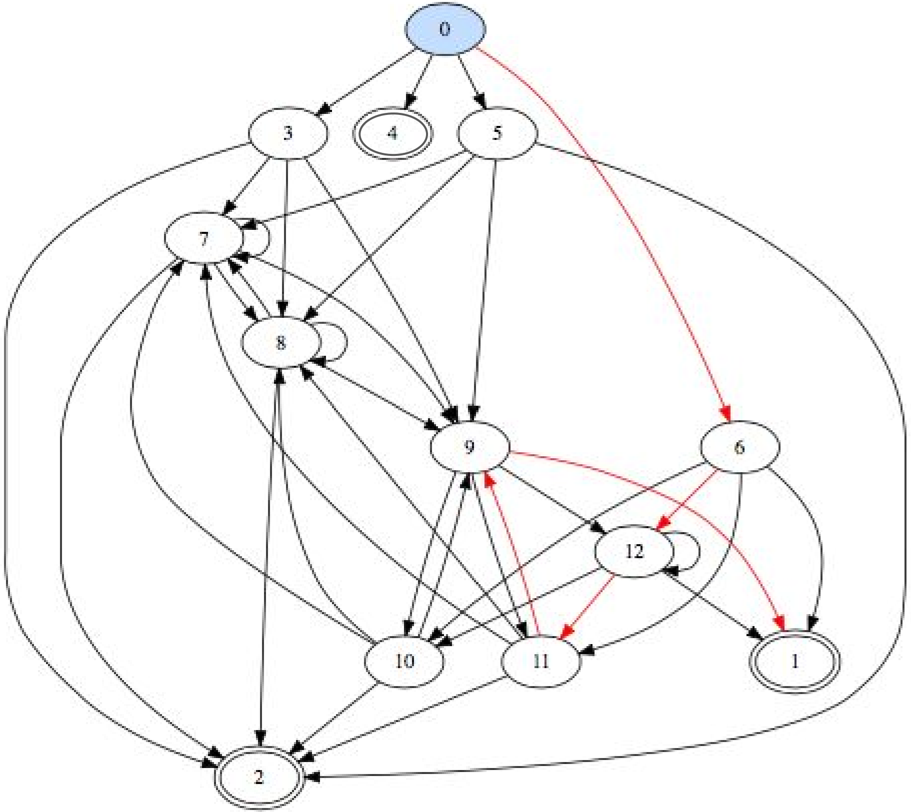
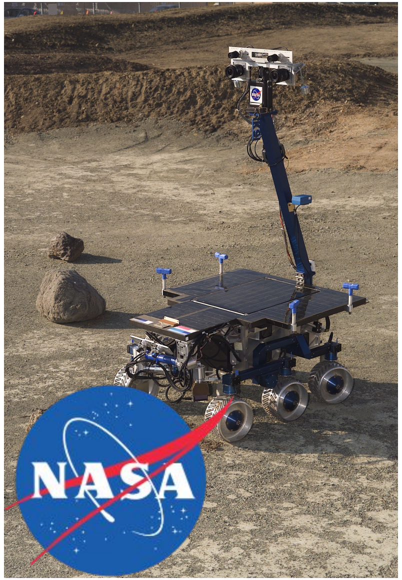


Tableau for always((next *a*) or *b*)



Node	Contents
Q_0	$(\{\Diamond \text{end}, \Box(\bigcirc a \vee b)\}, \emptyset)$
Q_1	$(\{b, \top, \bigcirc a \vee b, \neg \bigcirc a, \text{end}, \Box(\bigcirc a \vee b)\}, \{\bot, a, \bigcirc \top \vee \bot, \bigcirc a, \bigcirc \top, \Box \neg \text{end}\})$
Q_2	$(\{a, b, \top, \bigcirc a \vee b, \neg \bigcirc a, \text{end}, \Box(\bigcirc a \vee b)\}, \{\bot, \bigcirc \top \vee \bot, \bigcirc a, \bigcirc \top, \Box \neg \text{end}\})$
Q_3	$(\{\bigcirc a \vee b, \bigcirc \top \vee \bot, \Diamond \text{end}, \bigcirc a, \bigcirc \top, \Box(\bigcirc a \vee b)\}, \{\bot, b, \neg \bigcirc a, \text{end}, \Box \neg \text{end}\})$
Q_4	$(\{b, \bigcirc a \vee b, \neg \bigcirc a, \text{end}, \Diamond \text{end}, \Box(\bigcirc a \vee b)\}, \{\bot, \bigcirc \top \vee \bot, \bigcirc a, \bigcirc \top, \Box \neg \text{end}\})$
Q_5	$(\{b, \bigcirc a \vee b, \bigcirc \top \vee \bot, \Diamond \text{end}, \bigcirc a, \bigcirc \top, \Box(\bigcirc a \vee b)\}, \{\bot, \neg \bigcirc a, \text{end}, \Box \neg \text{end}\})$
Q_6	$(\{b, \bigcirc a \vee b, \bigcirc \top \vee \bot, \neg \bigcirc a, \Diamond \text{end}, \bigcirc \top, \Box(\bigcirc a \vee b)\}, \{\bot, \text{end}, \bigcirc a, \Box \neg \text{end}\})$
Q_7	$(\{a, \top, \bigcirc a \vee b, \bigcirc \top \vee \bot, \bigcirc a, \bigcirc \top, \Box(\bigcirc a \vee b)\}, \{\bot, b, \neg \bigcirc a, \text{end}, \Box \neg \text{end}\})$
Q_8	$(\{a, b, \top, \bigcirc a \vee b, \bigcirc \top \vee \bot, \bigcirc a, \bigcirc \top, \Box(\bigcirc a \vee b)\}, \{\bot, \neg \bigcirc a, \text{end}, \Box \neg \text{end}\})$
Q_9	$(\{a, b, \top, \bigcirc a \vee b, \bigcirc \top \vee \bot, \neg \bigcirc a, \bigcirc \top, \Box(\bigcirc a \vee b)\}, \{\bot, \text{end}, \bigcirc a, \Box \neg \text{end}\})$
Q_{10}	$(\{\top, \bigcirc a \vee b, \bigcirc \top \vee \bot, \bigcirc a, \bigcirc \top, \Box(\bigcirc a \vee b)\}, \{\bot, a, b, \neg \bigcirc a, \text{end}, \Box \neg \text{end}\})$
Q_{11}	$(\{b, \top, \bigcirc a \vee b, \bigcirc \top \vee \bot, \bigcirc a, \bigcirc \top, \Box(\bigcirc a \vee b)\}, \{\bot, a, \neg \bigcirc a, \text{end}, \Box \neg \text{end}\})$
Q_{12}	$(\{b, \top, \bigcirc a \vee b, \bigcirc \top \vee \bot, \neg \bigcirc a, \bigcirc \top, \Box(\bigcirc a \vee b)\}, \{\bot, a, \text{end}, \bigcirc a, \Box \neg \text{end}\})$

Motivation

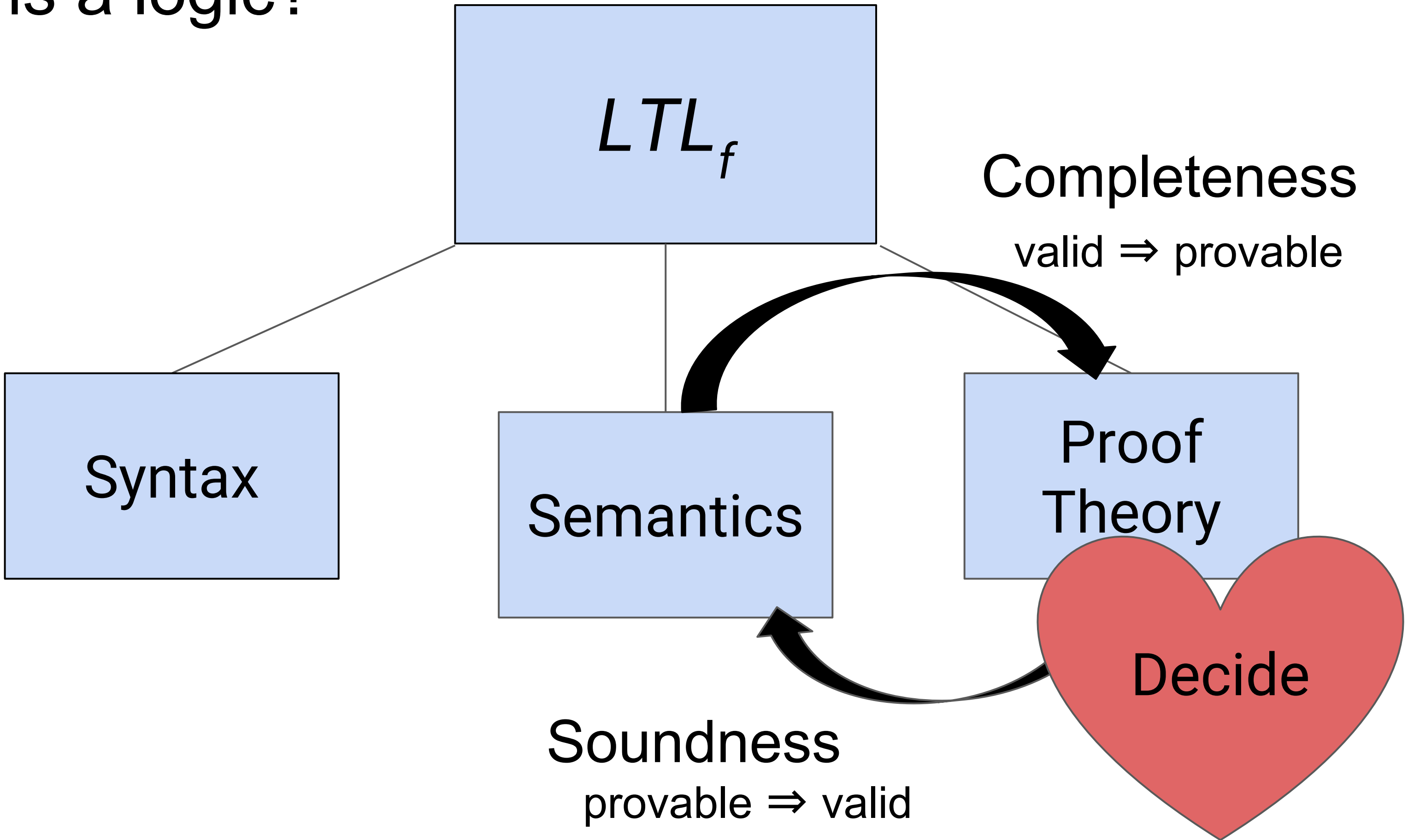


LTL

Motivation for my work

NetKAT + LTL_f = Temporal NetKAT

What is a logic?



Questions?

Proof Theory

Tauts in Classical Logic

$\vdash \text{next}\,(a \rightarrow b) \;\equiv\; \text{next}(a) \rightarrow \text{next}(b)$

$\vdash \text{end} \rightarrow \neg\, \text{next}(a)$

$\vdash \text{ever(end)}$

$\vdash \text{ever}\; a \;\equiv\; a \vee \text{next}(\text{ever}(a))$

If $\vdash a$, then $\vdash \text{wk\_next}(a)$

If $\vdash a \rightarrow b$

and $\vdash a \rightarrow \text{wk\_next}(a)$,

then $\vdash a \rightarrow \text{always}(b)$

Full Semantics

$$K_i^n(v) = \eta_i(v)$$

$$K_i^n(\bot) = \textbf{false}$$

$$K_i^n(a \to b) = \begin{cases} \textbf{true} & \text{if } K_i^n(a) = \textbf{false} \\ \textbf{true} & \text{if } K_i^n(b) = \textbf{true} \\ \textbf{false} & \text{otherwise} \end{cases}$$

$$K_i^n(\bigcirc a) = \begin{cases} K_{i+1}^n(a) & \text{if } i < n \\ \textbf{false} & \text{otherwise} \end{cases}$$

$$K_i^n(a \,\mathcal{W}\, b) = \begin{cases} \textbf{true} & \text{if } K_j^n(a) = \textbf{true}, \text{ for all } i \le j \le n \\ \textbf{true} & \text{if there exists } i \le k \le n, \text{ such that } K_k^n(b) = \textbf{true} \\ & \text{ and for every } j \text{ such that } i \le j < k, K_i^n(a) = \textbf{true} \\ \textbf{false} & \text{otherwise} \end{cases}$$

Syntactic Sugar

$\text{always}(a) \equiv a \text{ wk\_until False}$

$\text{ever}(a) \equiv \neg \text{ always}(\neg a)$

$a \text{ until } b \equiv a \text{ wk\_until } b \wedge \text{ever}(b)$

$\text{wk\_next (a)} \equiv \neg \text{ next}(\neg \text{a})$

$\text{end} \equiv \neg \text{ next(True)}$

Linear Temporal Logic over finite traces

Unique Successor

next, always, ever, until, etc.

$\vdash$ ever end

$T^n = (f_1, \ldots, f_n)$